TWO-POINT BOUNDARY VALUE PROBLEMS FOR INTEGRABLE EVOLUTION EQUATIONS IN ONE SPACE DIMENSION

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1. INTRODUCTION

The inverse scattering, or inverse spectral, transform is a method of solving the initial value problem for a class of nonlinear equations in one space and one time variable; the equations in this class are called *integrable*. This method, first proposed in [8] for the Korteweg-deVries (KdV) equation, has then been successfully applied to many other integrable nonlinear evolution equations. It provides an explicit integral representation of the solution, and it is based on the Lax property of integrable equations, that is on the fact that integrable equations can be written as the compatibility condition of two linear ordinary differential equations (ODE), one in the spatial variable x and one in the temporal variable t; these two equations are called a *Lax pair*. The classical inverse spectral method uses the spectral analysis of the x part of the Lax pair, while the t part is used to determine the time evolution of the spectral data, i.e. of the data that appear in the integral representation of the solution of the given problem.

For over twenty years, after the discovery of the inverse scattering method, the most outstanding open problem in this field has been the solution of initial and *boundary* value problem for the important class of integrable evolution equations; to this class belong in particular several equations modelling water wave propagation, such as the KdV, the Benjamin-Ono (BO), the sine-Gordon and the nonlinear Schrödinger equations, all of which are of great relevance for physical applications. A first step in this direction was the understanding [4] that for the inverse scattering method to yield a spectrally decomposed integral representation of the solution, one needs to perform the spectral analysis not only of one, but rather of both equations in the Lax pair. The second crucial observation, due to Gelfand and Fokas [3], is that linear partial differential equations (PDE) can be regarded as a special case of integrable equations, and in particular possess a Lax pair formulation; thus linear evolution PDE's can be solved by the inverse spectral method. These two facts have been jointly exploited by Fokas, who also made the important discovery that performing the *simultaneous* spectral analysis of the two equations in the Lax pair greatly simplifies the analysis and clarifies the meaning of each step; based on this, in [2], he presents a general approach to solving boundary value problems on the half-line for some particular examples of linear and of integrable nonlinear PDE's. This approach uses the fact that the spectral analysis of the nonlinear integrable case can be based on the analogous analysis of the associated *linearized* equation. Following this important work, several different boundary value problems have been studied for integrable PDE's, including problems involving time-dependent geometries [6]; moreover, the method has been applied to investigate many issues arising in physical applications.

The aim of this article is to use the general approach of [2] and its application to linear evolution equations on the half-line, [5, 7], to study two-point boundary value problems for integrable evolution equations in one space variable. We shall focus on the linear dispersive case, i.e. on the problem of determining the function $q(x,t)$ that for given appropriate initial and boundary conditions, satisfies the equation

$$\begin{aligned}&(\partial_t + i\omega(-i\partial_x))\, q(x,t) = 0,\\ &0 \le x \le L,\ 0 \le t \le T,\end{aligned} \tag{1}$$

where L is a given real number, and $\omega(k) = \sum_{j=0}^{n} \alpha_j k^j$, $\alpha_j \in \mathbf{R}$, is the dispersion relation.

For domains with a fixed geometry, as is the case for the two-point boundary we consider presently, the method consists of the following steps: (a) Formulating the given PDE as the compatibility condition of two linear eigenvalue ODE's. (b) Performing the simultaneous spectral analysis of these two equations, i.e. constructing a function $\mu(x,t,k)$ which solves both eigenvalue equations and which for (x,t) in the given domain, is bounded for all complex k, where k is the spectral parameter of the eigenvalue equations. In fact, for the present case, the function $\mu(x,t,k)$ is actually sectionally analytic in k. This means that there exists an oriented contour Γ dividing the complex k plane in two multiply-connected regions, a $(+)$ and $(-)$ region (by convention, the $(+)$ region lies to the left of the positive orientation), such that μ is analytic in each of these regions. This leads to the formulation of a classical mathematical problem, known as a Riemann-Hilbert (RH) problem, on the fixed contour Γ of the complex k-plane; the unique solution of this problem yields a spectral representation of the solution of the equation, in terms of a function $\rho(k)$ which we call the *spectral data*. (c) Identifying all global relations satisfied by the boundary values of the solution.

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These three steps are algorithmic, and only step (c) makes reference to the boundary conditions. One can then use the relations of (c) to determine how many boundary conditions must be prescribed in order for the problem to be well posed. For such a well posed problem, one then has to compute the spectral data $\rho(k)$ involved in the representation of the solution, in terms only of the given initial and boundary data. This last step depends in an essential way on the particular domain and and boundary conditions one is considering, and each case requires a separate analysis.

Step (a) and (b) can be performed without essential differences in the integrable nonlinear case as well; the difference between linear and nonlinear behaviour emerges essentially in step (c), as in this case the global relations one obtains are nonlinear as opposed to linear; thus their resolution is much harder and does not lead to explicit formulas.

This program has been carried out for linear evolution PDE's on the half line, and for several particular examples of nonlinear integrable PDE's on various fixed or time-dependent domains. In the linear case, the solution has a purely integral representation, whereas in the integrable nonlinear case, this integral representation is supplemented by a discrete sum, whose terms correspond to the celebrated class of special solutions called *solitons*. The difference introduced by the two-point boundary is that even for the linear case, the representation of the solution consists not only of an integral along a specified complex contour, but also of a discrete sum, taken on the zeroes of a certain discriminant function $D(k)$ which depends only on the particular spatial interval. Physically, these zeroes correspond to the resonant modes one expects to be present for such a one-dimensional constrained evolution problem.

In Section 2 we outline the result for the general linear equation (1), and illustrate the details for the particular case of the two-point boundary value problem for the linearized KdV equation $q_t + q_{xxx} = 0$. In Section 3 we indicate how the results for the linear case can be used to solve the same problem for the KdV equation $q_t + q_{xxx} + 6qq_x = 0$.

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2. LINEAR EVOLUTION EQUATIONS

Rather than considering the general linear problem (1), we restrict attention only to dispersive evolution equations corresponding to the dispersion relation $\omega(k) = \pm(-1)^n k^n$, i.e. of the form

$$\begin{aligned} q_t \pm (i)^{n+1}\partial_x^n q = 0, &\qquad 0 \le x \le L, \\ q(x,0) = q(x), &\qquad 0 \le t \le T. \end{aligned} \tag{2}$$

where $q(x)$ is a given smooth function, and boundary conditions are imposed in such a way that the problem has a unique solution. This simplification does not imply a conceptual loss of generalization; indeed, it is shown in [7] that the analysis of the general problem (1) is essentially based on the analysis of the equation obtained by considering only the leading order x-derivative. To fix ideas, we consider below only the choice of $+$ sign in equation (2); we remark shortly on the choice of $-$ sign at the end of this section.

The first step in our method, the expression of the equation as the compatibility of a pair of ODE's, can be achieved algorithmically [2]; it is easy to check that a Lax pair for this equation is given by

$$\mu_x - ik\mu = q, \tag{3}$$

$$\mu_t + (-1)^{n+1} i k^n \mu = \tilde{q}, \tag{4}$$

where $\mu = \mu(x,t,k)$ and

$$\tilde{q} = i^{n-1}\left[q_{n-1} + ikq_{n-2} \ldots + (ik)^{n-1} q_0\right],$$

$$q_j(x,t) = \partial_x^j q(x,t)$$

The second step involves finding a solution $\mu(x,t,k)$, bounded for all k in the complex plane, of both equations (3) and (4). Consider

$$\begin{aligned} \mu_1(x,t,k) &= \int_0^x \mathrm{e}^{ik(x-y)} q(y,t)dy \\ &+\mathrm{e}^{ikx}\int_0^t \mathrm{e}^{-i\omega(k)(t-s)}\tilde{q}(0,s,k)ds, \\ \mu_2(x,t,k) &= \int_0^x \mathrm{e}^{ik(x-y)} q(y,t)dy \\ &-\mathrm{e}^{ikx}\int_t^T \mathrm{e}^{-i\omega(k)(t-s)}\tilde{q}(0,s,k)ds, \\ \mu_3(x,t,k) &= -\int_x^L \mathrm{e}^{ik(x-y)} q(y,t)dy \\ &+\mathrm{e}^{ik(x-L)}\int_0^t \mathrm{e}^{-i\omega(k)(t-s)}\tilde{q}(L,s,k)ds, \\ \mu_4(x,t,k) &= -\int_x^L \mathrm{e}^{ik(x-y)} q(y,t)dy \\ &-\mathrm{e}^{ik(x-L)}\int_t^T \mathrm{e}^{-i\omega(k)(t-s)}\tilde{q}(L,s,k)ds. \end{aligned}$$

The boundedness of each of these functions depends on the negativity of the real part of the exponent of the various exponentials involved. Since e^{ikx} is bounded, for $x \ge 0$, only for k in the upper half plane $\mathbf{C}^+$, the only exponential one needs to analyze is $\mathrm{e}^{-i\omega(k)t}$. Let $D^+ = \{k : \mathrm{Im}(\omega(k)) \le 0\}$ and $D^- = \{k : \mathrm{Im}(\omega(k)) \ge 0\}$. Then if $D_1 = \mathbf{C}^+ \cap D^+$, $D_2 = \mathbf{C}^+ \cap D^-$, $D_3 = \mathbf{C}^- \cap D^+$, and $D_4 = \mathbf{C}^- \cap D^-$, each μ_i is bounded, and in addition analytic, in the corresponding D_i. Define $\mu = \mu_i$ for $k \in D_i$, and compute the jump $\mu_i - \mu_j$, $i \ne j$, of the function $\mu(x,t,k)$ across the boundaries of the various regions D_i; since $\mu_i - \mu_j$ satisfies the homogeneous version of equations (3)-(4), this jump is always of the form

$$\mu_i - \mu_j = \mathrm{e}^{ikx - i\omega(k)t}\rho(k),\ i \ne j, \tag{5}$$

and it is easy to compute that

$$\rho(k) = \begin{cases} \hat{q}(k) = \int_0^L \mathrm{e}^{-iky} q(y)dy, \\ \nu(0,k) = \int_0^T \mathrm{e}^{i\omega(k)s}\tilde{q}(0,s)ds, \\ \nu(L,k) = \int_0^T \mathrm{e}^{i\omega(k)s}\tilde{q}(L,s)ds. \end{cases} \tag{6}$$

Moreover, μ satisfies the asymptotic behaviour

$$\mu(x,t,k) = \frac{iq(x,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty.$$

This asymptotic estimate, the sectional analyticity of μ (i.e. the analyticity of each μ_i), and the jump (5) define a RH problem. This problem has a unique solution, expressed as the Cauchy integral of the jump of $\mu(x,t,k)$ along the contour in $\mathbf{C}$ across which μ has a jump. In particular, the variables x and t dependence of this solution is explicit, because in the jump function (5) x and t appear only exponentially. This unique solution together with equation (3) yields for $q(x,t)$ the desired spectrally decomposed integral representation, in terms of the spectral data $\hat{q}(k)$, $\nu(0,k)$ and $\nu(L,k)$; these spectral data are defined only in terms of the initial and boundary data, i.e. of the functions $q(x)$ and $q_j(0,t)$, $j = 0, \ldots, n-1$.

In general not all of the boundary conditions $q_j(t)$ can be prescribed independently. However, to achieve the third step of the program, one notes that all the $\mu_i(x,t,k)$ have a second representation, which can be used to obtain a set of global relations that the spectral data must satisfy. After some manipulations, these global relations determine a system of algebraic equations for the boundary conditions; according to the number of independent equations in this system, one can determine how many independent boundary conditions need be prescribed in order for the problem to be well posed. In the present case of a two-point boundary value problem, in order to solve the above system, one needs to divide by a certain determinant $D(k)$; the residue theorem implies that, at the zeros of this determinant, the integral representation involving $1/D(k)$ must be supplemented by a discrete contribution extended over these zeros.

Rather than performing this analysis in general, we now give the precise result and some details for the equation $q_t + q_{xxx} = 0$. The detailed analysis of this equation can be found in [1].

Proposition 1 *Let $q(x,t)$ satisfy*

$$\begin{aligned} &q_t + q_{xxx} = 0, && 0 \le x \le L,\ 0 \le t \le T, \\ &q(x,0) = q(x), && q(x) \in \mathbf{C}^3([0,L]). \end{aligned} \tag{7}$$

Suppose that appropriate boundary conditions are given at $x = 0$ and $x = L$ such that there exists a unique global solution $q(x,t) \in \mathbf{C}^3[0,L] \times \mathbf{C}^1[0,T]$.

Define, in the complex k-plane, the six regions $\{j\}_{j=1,\ldots,6}$, where $k \in \{j\}$ if

$$[(j-1) * \pi]/3 \le \arg(k) \le [j * \pi]/3,$$

and denote by $\partial\{j\}$ their oriented boundary, see Figure 1.

Then this unique solution is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{e}^{ikx+ik^3t}\hat{q}(k)dk \\ &+ \int_{\partial\{2\}} \mathrm{e}^{ikx+ik^3t}\nu(0,k)dk \\ &+ \int_{\partial\{4\}\cup\partial\{6\}} \mathrm{e}^{ik(x-L)+ik^3t}\nu(L,k)dk \Bigg\}, \end{aligned} \tag{8}$$

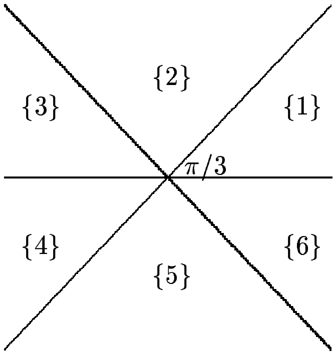

Figure 1: The regions $\{1\}, \ldots, \{6\}$ associated with equation (7)

where the entire functions $\hat{q}(k)$ and $\nu(w,k)$ are defined in (6). Moreover, for $\arg(k) \in [\pi/3, 2\pi/3]$, the following global conditions are valid

$$\begin{aligned} &\nu(L,k) - \mathrm{e}^{ikL}\nu(0,k) + \mathrm{e}^{ikL}\hat{q}_0(k) \\ &= \mathrm{e}^{ikL-ik^3T}\hat{q}_T(k), \\ &\mathrm{e}^{-i\zeta kL}\nu(L,\zeta k) - \nu(0,\zeta k) + \hat{q}_0(\zeta k) \\ &= \mathrm{e}^{-ik^3T}\hat{q}_T(k), \\ &\mathrm{e}^{-i\zeta^2 kL}\nu(L,\zeta^2 k) - \nu(0,\zeta^2 k) + \hat{q}_0(\zeta^2 k) \\ &= \mathrm{e}^{-ik^3T}\hat{q}_T(k), \end{aligned} \tag{9}$$

where $\zeta = \mathrm{e}^{-2\pi i/3}$ and

$$\hat{q}_T(k) = \int_0^L \mathrm{e}^{-ikx}q(x,T)dx. \tag{10}$$

To obtain the representation (8), we notice that in this case μ_1 is bounded in $\{1\} \cup \{3\}$, μ_2 in $\{2\}$, μ_3 in $\{5\}$ and μ_4 in $\{4\} \cup \{6\}$ (see Figure 2); thus their differences satisfy

$$\begin{aligned} \mu_1 - \mu_4 &= \mathrm{e}^{ikx+ik^3t}\left[\mathrm{e}^{-ikL}\nu(L,k)\right. \\ &\quad \left.+\hat{q}(k)\right],\ k \in \mathbf{R}, \\ \mu_1 - \mu_2 &= \mathrm{e}^{ikx+ik^3t}\nu(0,k), \\ &\quad k \in \partial\{2\}, \\ \mu_3 - \mu_4 &= \mathrm{e}^{ik(x-L)+ik^3t}\nu(L,k), \\ &\quad k \in \partial\{5\}, \end{aligned} \tag{11}$$

Moreover, μ_3 has the second representation

$$\begin{aligned} \mu_3(x,t,k) &= \mathrm{e}^{ik^3t}\mu_3(x,0,k) \\ &+ \int_0^t \mathrm{e}^{ik^3(t-s)}\tilde{q}(x,s,k)ds. \end{aligned}$$

Evaluating this equation at $x = 0$, $t = T$ and using the the first representation of $\mu_3(x,t,k)$, we obtain, for $k \in \{5\}$, the global relation

$$\begin{aligned} &\mathrm{e}^{-ikL+ik^3T}\nu(L,k) - \mathrm{e}^{ik^3T}\nu(0,k) \\ &+\mathrm{e}^{ik^3T}\hat{q}(k) = \hat{q}_T(k), \end{aligned} \tag{12}$$

where $\hat{q}$ is defined in (6) and $\hat{q}_T(k)$ is given by (10). We note that multiplying the global relation (12) by appropriate exponentials, we obtain the analogous relations for all

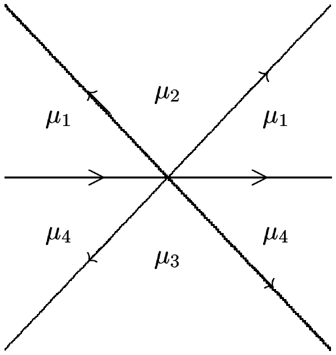

Figure 2: The set-up for the Riemann-Hilbert problem associated with equation (7)

k in $\mathbf{C}$: then by rotation (i.e. multiplication by the factor ζ or ζ^2, where $\zeta = \mathrm{e}^{-2\pi i/3}$), the global relations in $\{4\}$ and $\{6\}$ yield two equations in $\{2\}$. Thus we obtain that in the region $\{2\}$ the three relations (9) hold. Since the boundary data are six, three at $x = 0$ and three at $x = L$, this means that we have a system of three independent algebraic equations in six independent variables; thus to have any hope of solvability, we must prescribe three of these variables. However, it turns out that this system can be solved in terms of $q(x, 0)$ and the three known boundary conditions only in some cases. This is due to the presence of the unknown term $\hat{q}_T(k)$; we can solve the system explicitly *only* in the case that this term, divided by the discriminant of the system, and multiplied by the exponential e^{ikx+ik^3t}, is bounded and analytic in the region $\{2\}$, so that its integral along the boundary of this region gives no contribution to the representation of the solution. For the present example, one can prove the following result.

Theorem 2 *The initial boundary value problem for (7) is well posed if any one of the functions $\partial_x^j q(0, t)$ and any two of the functions $\partial_x^j q(L, t)$, $j = 0, 1, 2$ are given. The unique solution of such a well posed problem is given by*

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{ikx+ik^3t}\hat{q}(k)dk\right.$$

$$+ \int_{\partial\{2\}} \mathrm{e}^{ikx+ik^3t}\frac{N(0,k}{D(k)})dk$$
$$+ \int_{\partial\{4\}} \mathrm{e}^{ik(x-L)+ik^3t}\frac{N(L,\zeta^2k)}{D(\zeta^2k)}dk$$
$$\left.+ \int_{\partial\{6\}} \mathrm{e}^{ik(x-L)+ik^3t}\frac{N(L,\zeta k)}{D(\zeta k)}dk\right\}$$
$$- \sum_{j=1}^{\infty} \mathrm{e}^{i\lambda_j^3t+i\lambda_jx}\frac{N(0,\lambda_j)+2N(L,\lambda_j)}{D'(\lambda_j)}.$$

Here, $\zeta = \mathrm{e}^{-2\pi i/3}$, $D(k)$ is given by

$$D(k) = (1-\zeta)(\zeta + \mathrm{e}^{ik(1-\zeta^2)L} + \zeta^2\mathrm{e}^{ik(1-\zeta)L}),$$

λ_j are the zeros of $D(k)$ in the region $\{2\}$. The terms $N(0, k)$, $N(L, k)$ can be computed explicitly in terms of the given boundary data.

To give a specific example,suppose that the given boundary data are $q(0, t) = q_0^0(t)$, $q(L, t) = q_0^L(t)$ and $q_x(L, t) = q_1^L(t)$; then the spectral data are given explicitly by the formulas

$$\begin{aligned}N(0,k) = (1 - EE_1)\hat{q}(\zeta^2k)+\\ k^2\nu_0(0,k)\left((\zeta-1)(\zeta+EE_1+\zeta^2EE_2)\right)\\ -(1-\zeta)k\nu_1(L,k)\left(E_1+\zeta^2E_2+\zeta E^2\right)\\ -(1-\zeta)k^2\nu_0(L,k)\left(E_2+\zeta^2E_1+\zeta E^2\right)\\ -(E_2-E_1)E\hat{q}(k)-(1-EE_2)\hat{q}(\zeta k)\end{aligned}$$

$$\begin{aligned}N(L,k) = +3Ek^2\nu_0(0,k)\\ -k\nu_1(L,k)(1+\zeta^2EE_1+\zeta EE_2)\\ -k^2\nu_0(L,k)(1+EE_1+EE_2)\\ +E(\hat{q}(k)+\zeta\hat{q}(\zeta k)+\zeta^2\hat{q}(\zeta^2k)),\end{aligned}$$

where, if $u = 0$ or $u = L$,

$$\begin{aligned}&\nu_0(u,k) = \int_0^T \mathrm{e}^{-ik^3t}q_0^u(t)dt,\\ &\nu_1(L,k) = \int_0^T \mathrm{e}^{-ik^3t}q_1^L(t)dt,\\ &E = \mathrm{e}^{ikL},\ E_1 = \mathrm{e}^{-i\zeta kL},\\ &E_2 = \mathrm{e}^{-i\zeta^2kL},\ F = \mathrm{e}^{-ik^3T}.\end{aligned}$$

We conclude this section with a remark on the equation obtained by choosing the $-$ sign in (2), i.e. we consider the same boundary value problem for the equation $q_t - q_{xxx} = 0$. In this case, we obtain a well posed problem if *two* conditions are given at $x = 0$ and *one* at $x = L$. This result can be proven by the same method outlined above; however it is an immediate consequence of the fact that this choice of sign corresponds to an inversion of the direction of the time variable.

3. INTEGRABLE NONLINEAR CASE: THE KDV EQUATION

We can now use the knowledge of the linearized equation solved in the previous section to find a representation of the solution of the following boundary value problem for the KdV equation:

$$\begin{aligned}q_t + q_{xxx} + 6qq_x = 0, &\qquad 0 \le x \le L,\\ q(x,0) = q(x), &\qquad 0 \le t \le T.\end{aligned}$$

We assume that boundary conditions are given in such a way that there exists a unique solution of the problem. This is possible as the KdV equation is integrable, and as such it admits a Lax pair representation. Although in the nonlinear case there is no algorithm to construct the Lax pair, in this case it has been known for a long time that the Lax pair is given by (see [4])

$$\begin{aligned}&\mu_x + ik[\sigma_3,\mu] = Q\mu,\\ &\mu_t + 4ik^3[\sigma_3,\mu] = \tilde{Q}\mu,\end{aligned}\tag{13}$$

where $[\cdot,\cdot]$ denotes the usual matrix commutator, $\mu(x,t,k)$ is a 2×2 matrix-valued function and the 2×2 matrices $Q(x,t,k)$ and $\tilde{Q}(x,t,k)$ are defined by

$$\begin{aligned} Q &= \frac{q}{2k}\left(\sigma_2 - i\sigma_3)\right), \\ \tilde{Q} &= 2kq\sigma_2 + q_x\sigma_1 + \frac{1}{2k}(q_{xx}+2q^2)\left(i\sigma_3-\sigma_2)\right), \end{aligned}$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The construction of the representation of the solution is analogous to the one carried out for the linear case, i.e. it is based on the simultaneous spectral analysis of the Lax pair. To construct the sectionally analytic function μ in this case, we proceed as follows: using the notation $\hat{\sigma}_3 A = [\sigma_3, A]$, then $\mathrm{e}^{\hat{\sigma}_3}A = \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$, matrix-valued solutions of both equations (13) are given by

$$\begin{aligned} M_1(x,t,k) = I + \int_0^x \mathrm{e}^{ik(x-y)\hat{\sigma}_3}(QM_1)(y,t,k)dy \\ +\mathrm{e}^{ikx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-i\omega(k)(t-s)\hat{\sigma}_3}(\tilde{Q}M_1)(0,s,k)ds, \end{aligned}$$

and by M_2, M_3 and M_4 defined similarly, in analogy with μ_2, μ_3 and μ_4. By analyzing where the column vectors of these functions are bounded and analytic, one arrives at a well defined RH problem, that is now matrix-valued rather than scalar, with jumps across the same contour as the RH problem associated with the linearized problem, examined in the previous section. The solution of this matrix-valued RH problem cannot be found in closed form, but is expressed through a linear integral equation of the Fredholm type; however, the x and t dependence of the solution is the same found for the linear problem, thus it is still explicit. The solution of the boundary value problem is then given by

$$q(x,t) = 2i\partial_x\left(\lim_{k\to\infty}\mu_{12}(x,t,k)\right),$$

where $\mu_{12}(x,t,k)$ indicates the $(1,2)$ component of the matrix $\mu(x,t,k)$.

Matrix RH problems, as opposed to scalar ones, can have singularities; these singularities yield a discrete contribution to the solution representation. However, this contribution differs from the discrete contribution found in the linear case, which was due purely to the form of the boundary of the domain. This kind of discrete contribution will be supplemented, in the present nonlinear case, by a discrete contribution due to the singularities of the RH problem. These singularities give rise to soliton solutions, which are very special solutions associated only with integrable equations.

Finally, we remark that the most difficult part of the analysis is now the resolution of the global relations, as these relations are now nonlinear; these global relations yield a system of nonlinear integral equations of the Volterra type for the boundary data, which one must analyze in order to prove which problems are well-posed. However, at least for small data (or small time intervals), the well-posedness of the linearized problem implies the well-posedness of the corresponding problem for the nonlinear equation, and we expect that the analysis of the nonlinear relations, although more complicated in the present nonlinear case, will yield the same results as in the linear one.

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