ON THE KAUFFMAN BRACKET OF PRETZEL LINKS

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ABSTRACT

In this note we give a closed-form formula for the Kauffman bracket of pretzel links. In particular this formula allows us to calculate the span of the Jones polynomial of any pretzel link (compare to partial results obtained in [1]).

Keywords: *Pretzel link, Kauffman bracket, Jones polynomial, span.*

1. KAUFFMAN BRACKET OF PRETZEL LINKS

In [1], Hara, Tani and Yamamoto calculated the span of the Jones polynomial of some pretzel links. In this note I give a simple formula for the Kauffman bracket of any pretzel link and deduce the span of the Jones polynomial of any pretzel link, improving clearly the results in the paper above mentioned.

The proofs are quite simple, requiring basically a good organization in carrying the calculations and manipulating the skein relations.

The interest of the paper is to provide the span of a large family of Jones polynomials, other than those of torus links. This has already been used to compare this number with some bounds in connection with the Bennequin number.

Given any integers $a_1, \ldots, a_n$, we denote by $P(a_1, \ldots, a_n)$ the pretzel link diagram shown in Figure 1. Here a_i indicates $|a_i|$ crossings, with "signs" $a_i/|a_i|$ if $a_i \neq 0$.

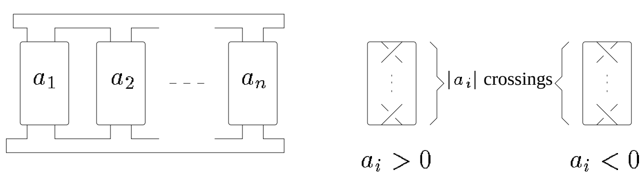

Figure 1: Pretzel link diagram $P(a_1, \ldots, a_n)$.

For a link diagram D we denote by $\langle D \rangle$ its Kauffman bracket with normalization $\langle \bigcirc \rangle = \delta = -A^{-2} - A^{2}$ (see [2]). Recall that $\langle D \rangle$ is a regular isotopy invariant of diagrams, defined by the following additional relations:

(i) $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)\,(\rangle$ (main skein relation).

(ii) $\langle D \sqcup \bigcirc \rangle = \delta \langle D \rangle$.

Here $\bigcirc$ is the diagram of the unknot with no crossings. In (i) the formula refers to three link diagrams that are exactly the same except near a point where they differ in the way indicated. In (ii) $D \sqcup \bigcirc$ is a diagram consisting of the diagram D together with an extra closed curve $\bigcirc$ that contains no crossings at all, not with itself nor with D. From these relations we can deduce the effect on $\langle D \rangle$ of a type I Reidemeister move on D:

(iii) $\langle \text{kink} \rangle = -A^{3} \langle \smile \rangle$

(iii') $\langle \text{kink} \rangle = -A^{-3} \langle \smile \rangle$

Definition. Let $[0] = 0$, and for any integer $a \neq 0$, let $[a] \in \mathrm{Z}[A^{\pm 1}]$ be defined by

$$[a] = A^{2\frac{a}{|a|} - 3a} \sum_{j=0}^{|a|-1} (-1)^{a+j+1} A^{4j\frac{a}{|a|}}.$$

Note that $[a]$ is a Laurent polynomial in the variable A. Its behaviour reminds the quantum integers. Clearly $[1] = A^{-1}$, $[-1] = A$, and we have the recurrence formulas

$$[a] = A[a-1] + A^{-1}(-A^{-3})^{a-1} \text{ if } a > 0,$$

$$[a] = A^{-1}[a+1] + A(-A^{3})^{-a-1} \text{ if } a < 0.$$

In addition we have the following equality (the proof is an easy exercise):

Lemma 1. $\delta[a] = -A^{a} + (-A^{-3})^{a}$.

Lemma 2. $\langle P(\ldots, a_{i-1}, a, a_{i+1}, \ldots) \rangle$
$= A^{a} \langle P(\ldots, a_{i-1}, 0, a_{i+1}, \ldots) \rangle + [a] \langle P(\ldots, a_{i-1}, a_{i+1}, \ldots) \rangle$.

Proof. If $a = 0$ the formula is trivial since $[0] = 0$ by definition. Let $a > 0$. We use induction on a. The case $a = 1$ follows from (i) and the equality $[1] = A^{-1}$. Assume the result in the cases $1, \ldots, a-1$. Then we have that

$$\begin{aligned}
&\langle P(\ldots, a_{i-1}, a, a_{i+1}, \ldots) \rangle \\
&= A \langle P(\ldots, a_{i-1}, a-1, a_{i+1}, \ldots) \rangle \\
&\quad + A^{-1}(-A^{-3})^{a-1} \langle P(\ldots, a_{i-1}, a_{i+1}, \ldots) \rangle \\
&\quad \text{(by (i) and (iii'))} \\
&= A(A^{a-1} \langle P(\ldots, a_{i-1}, 0, a_{i+1}, \ldots) \rangle \\
&\quad + [a-1] \langle P(\ldots, a_{i-1}, a_{i+1}, \ldots) \rangle) \\
&\quad + A^{-1}(-A^{-3})^{a-1} \langle P(\ldots, a_{i-1}, a_{i+1}, \ldots) \rangle \\
&\quad \text{(by hypothesis of induction)}
\end{aligned}$$

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$$
\begin{aligned}
&= A^a\langle P(\ldots, a_{i-1}, 0, a_{i+1}, \ldots)\rangle \\
&\quad + (A[a-1] + A^{-1}(-A^{-3})^{a-1})\langle P(\ldots, a_{i-1}, a_{i+1}, \ldots)\rangle \\
&= A^a\langle P(\ldots, a_{i-1}, 0, a_{i+1}, \ldots)\rangle \\
&\quad + [a]\langle P(\ldots, a_{i-1}, a_{i+1}, \ldots)\rangle,
\end{aligned}
$$

the last equality since $[a] = A[a-1] + A^{-1}(-A^{-3})^{a-1}$ if $a > 0$.

For $a < 0$ we use induction on $|a|$. The case $a = -1$ follows from (i) and the equality $[-1] = A$. Assume the result in the cases $-1, \ldots, a+1$ and (exercise) proceed as in case $a > 0$, using (iii) instead of (iii'). □

Remark 1. Note that $\langle P(a)\rangle = A^a\langle P(0)\rangle + [a]\langle \bigcirc\bigcirc \rangle = A^a\delta + [a]\delta^2 = \delta(-A^{-3})^a$, the last equality by Lemma 1. Of course, this agrees with the fact that $P(a)$ is the trivial knot with $|a|$ kinks, negative kinks if $a > 0$, positive kinks if $a < 0$.

Lemma 3. $\langle P(\ldots, a, 0)\rangle = (A^a\delta + [a])\langle P(\ldots, 0)\rangle$.

Proof. We have that

$$
\begin{aligned}
&\langle P(\ldots, a, 0)\rangle \\
&\quad = A^a\langle P(\ldots, 0, 0)\rangle + [a]\langle P(\ldots, 0)\rangle \quad \text{(by Lemma 2)} \\
&\quad = A^a\delta\langle P(\ldots, 0)\rangle + [a]\langle P(\ldots, 0)\rangle \quad \text{(by (ii))} \\
&\quad = (A^a\delta + [a])\langle P(\ldots, 0)\rangle.
\end{aligned}
$$

□

Theorem 1. The Kauffman bracket of the pretzel link diagram $P(a_1, \ldots, a_n)$ is given by the formula

$$
\langle P(a_1, \ldots, a_n)\rangle = \prod_{i=1}^{n}(A^{a_i}\delta + [a_i]) + (\delta^2 - 1)\prod_{i=1}^{n}[a_i].
$$

Proof. By induction on n. If $n = 1$ then we have $\langle P(a)\rangle = \delta(-A^{-3})^a$ by Remark 1, and this is equal to $(A^a\delta + [a]) + (\delta^2 - 1)[a] = \delta(A^a + \delta[a])$ by Lemma 1.

Assume now the cases $1, \ldots, n-1$. We have that

$$
\begin{aligned}
&\langle P(a_1, \ldots, a_n)\rangle \\
&\quad = A^{a_n}\langle P(a_1, \ldots, a_{n-1}, 0)\rangle + [a_n]\langle P(a_1, \ldots, a_{n-1})\rangle \\
&\qquad \text{(by Lemma 2)} \\
&\quad = A^{a_n}(A^{a_{n-1}}\delta + [a_{n-1}])\langle P(a_1, \ldots, a_{n-2}, 0)\rangle \\
&\qquad + [a_n]\langle P(a_1, \ldots, a_{n-1})\rangle,
\end{aligned}
$$

the last equality by Lemma 3. Now we apply the hypothesis of induction and the fact that $[0] = 0$ to obtain the following:

$$
\begin{aligned}
&\langle P(a_1, \ldots, a_n)\rangle \\
&\quad = A^{a_n}(A^{a_{n-1}}\delta + [a_{n-1}])\delta\prod_{j=1}^{n-2}(A^{a_j}\delta + [a_j]) \\
&\qquad + [a_n]\Big(\prod_{k=1}^{n-1}(A^{a_k}\delta + [a_k]) + (\delta^2 - 1)\prod_{k=1}^{n-1}[a_k]\Big) \\
&\quad = \prod_{i=1}^{n}(A^{a_i}\delta + [a_i]) + (\delta^2 - 1)\prod_{i=1}^{n}[a_i].
\end{aligned}
$$

□

Remark 2. We will use the formula in Theorem 1 (presenting the Kauffman bracket of a pretzel link as the sum of two products) for obtaining the span of the Jones polynomial of any pretzel link. This formula can be obtained directly by using Lemma 2 and the fact that $\langle \bigcirc \rangle = \delta$. By applying Lemma 2 we obtain that

$$
\begin{aligned}
&\langle P(a_1, \ldots, a_n)\rangle \\
&= \sum_{I\subset\{1,\ldots,n\}}\Big(\prod_{i\in I}A^{a_i}\prod_{\substack{j\in\{1,\ldots,n\}\\ j\notin I}}[a_j]\langle P(b_1(I), \ldots, b_n(I))\rangle\Big)
\end{aligned}
$$

where $b_k(I) =][= 0$ if $k \in I$ and $b_k(I) = \asymp$ if $k \notin I$ (hence it has no effect on the diagram and can be deleted). Note that for a given $I \subset \{1, \ldots, n\}$, $I \neq \emptyset$, the link $P(b_1(I), \ldots, b_n(I))$ is nothing but the trivial link with $|I|$ components. Hence

$$
\begin{aligned}
&\langle P(a_1, \ldots, a_n)\rangle \\
&\quad = \sum_{\emptyset\neq I\subset\{1,\ldots,n\}}\Big(\delta^{|I|}\prod_{i\in I}A^{a_i}\prod_{j\in\{1,\ldots,n\}\setminus I}[a_j]\Big) \\
&\qquad + \prod_{i=1}^{n}[a_i]\langle P(\bigcirc\bigcirc)\rangle \\
&\quad = \prod_{i=1}^{n}(A^{a_i}\delta + [a_i]) - \prod_{i=1}^{n}[a_i] + \delta^2\prod_{i=1}^{n}[a_i] \\
&\quad = \prod_{i=1}^{n}(A^{a_i}\delta + [a_i]) + (\delta^2 - 1)\prod_{i=1}^{n}[a_i].
\end{aligned}
$$

Similarly, we denote by $\langle D\rangle_1$ the Kauffman bracket of the diagram D defined through the same relations (i) and (ii) but with normalization $\langle \bigcirc \rangle_1 = 1$. It follows that $\langle D\rangle = \delta\langle D\rangle_1$ for every link diagram D.

Remark 3. Note that $\langle P(a_1, \ldots, a_n)\rangle_1$ is given by the formula

$$
\sum_{k=1}^{n}\delta^{k-1}\Big(\sum_{\substack{I\subset\{1,\ldots,n\}\\ |I|=k}}\prod_{i\in I}A^{a_i}\prod_{j\in\{1,\ldots,n\}\setminus I}[a_j]\Big) + \delta\prod_{i=1}^{n}[a_i].
$$

To prove this, we check the relation $\delta\langle P(a_1, \ldots, a_n)\rangle_1 = \langle P(a_1, \ldots, a_n)\rangle$. But this is just the equality

$$
\begin{aligned}
&\sum_{k=1}^{n}\delta^{k}\Big(\sum_{\substack{I\subset\{1,\ldots,n\}\\ |I|=k}}\prod_{i\in I}A^{a_i}\prod_{j\in\{1,\ldots,n\}\setminus I}[a_j]\Big) + \delta^2\prod_{i=1}^{n}[a_i] \\
&\qquad = \prod_{i=1}^{n}(A^{a_i}\delta + [a_i]) + (\delta^2 - 1)\prod_{i=1}^{n}[a_i].
\end{aligned}
$$

2. SPAN OF THE JONES POLYNOMIAL OF PRETZEL LINKS

If L is an oriented link, we denote by $V(L)$ its Jones polynomial with normalization $V(\bigcirc) = -t^{-1/2} - t^{1/2}$ (see [2]). Recall that $V(L) = (-A)^{-3w(D)}\langle D\rangle$ after the substitution $A = t^{-1/4}$, where D is an oriented diagram of

L and $w(D)$ is its writhe. It follows that $span(\langle D\rangle) = 4span(V(L))$.

Remark 4. Recall that:

(1) For any permutation σ of $\{1,\dots,n\}$ the two link diagrams $P(a_1,\dots,a_n)$ and $P(a_{\sigma(1)},\dots,a_{\sigma(n)})$ are mutants of each other, and thus their Kauffman brackets agree. This also follows from the formula in Theorem 1.

(2) The span of the Kauffman bracket is unchanged under mirror image.

(3) Any consecutive pair $(1,-1)$ in $(a_1,\dots,a_n)$ can be cancelled via the type II Reidemeister move.

Definition. For a pretzel link diagram $P(a_1,\dots,a_n)$, we write r for the number of entries $a_i > 1$, s for the number of entries $a_i < -1$, z for the number of entries $a_i = 0$, α for the number of entries $a_i = 1$ and β for the number of entries $a_i = -1$. We also set $\lambda = \alpha - \beta$.

Theorem 2. Let $P(a_1,\dots,a_n)$ be an unoriented pretzel diagram of an oriented pretzel link L, with r, s, z and λ as in the above definition. Then we have the following values for the span of the Jones polynomial $V(L)$:

(i) $\sum_{|a_i|>1}|a_i| + z$ if $z>0$.

In the remaining cases we consider $z=0$.

(ii) $\sum_{|a_i|>1}|a_i| - \min\{1, r+\lambda, s-\lambda\}+1$ if $r+\lambda\neq 1$ and $s-\lambda\neq 1$.

(iii) $\sum_{|a_i|>1}|a_i| - 1$ if $r+\lambda=1$, $r>1$ and $s-\lambda\neq 1$.

(iv) Assume that $r+\lambda = 1$, $r=1$ and $s>1$. Assume in addition that $a_1 > 1$, $a_j < -1$ if $j\in\{2,\dots,n\}$ and $|a_2|\leq\dots\leq|a_n|$:

(a) $\sum_{|a_i|>1}|a_i| - \min\{a_1, |a_2|-1\}$ if $a_1\neq|a_2|-1$.

(b) $\sum_{|a_i|>1}|a_i| - \min\{|a_2|, |a_3|-1\}$ if $a_1=|a_2|-1$ and $|a_2|\neq|a_3|-1$.

(c) $\sum_{|a_i|>1}|a_i| - (|a_3|+1)$ if $a_1=|a_2|-1$, $|a_2|=|a_3|-1$ and $|a_3|<|a_4|-1$.

(d) $\sum_{|a_i|>1}|a_i| - |a_2|$ if $a_1=|a_2|-1$, $|a_2|=|a_3|-1$ and $|a_3|=|a_4|$.

(e) $\sum_{|a_i|>1}|a_i| - |a_3|$ if $a_1=|a_2|-1$, $|a_2|=|a_3|-1$ and $|a_3|=|a_4|-1$.

Proof. We will compute $span(V(L))$ by just calculating $span(\langle P(a_1,\dots,a_n)\rangle)$ and dividing by 4. We set $\varepsilon_i = (-1)^{a_i}$ for every $i\in\{a_1,\dots,a_n\}$. By Theorem 1 and Lemma 1 we have that

$$\delta^n\langle P(a_1,\dots,a_n)\rangle = \prod_{i=1}^n A^{-3a_i}\left(\prod_{i=1}^n((\delta^2-1)A^{4a_i}+\varepsilon_i) + (\delta^2-1)\prod_{i=1}^n(-A^{4a_i}+\varepsilon_i)\right)$$

and hence

$$4n + span(\langle P(a_1,\dots,a_n)\rangle) = span(\delta^n\langle P(a_1,\dots,a_n)\rangle).$$

Let $B = A^4$. Then $\delta^2 - 1 = B^{-1}+1+B$ and we have that

$$\begin{aligned} &span(\delta^n\langle P(a_1,\dots,a_n)\rangle)\\ &= span\left(\prod_{i=1}^n((\delta^2-1)A^{4a_i}+\varepsilon_i) + (\delta^2-1)\prod_{i=1}^n(-A^{4a_i}+\varepsilon_i)\right)\\ &= 4\ span_B\left(\prod_{i=1}^n((B^{-1}+1+B)B^{a_i}+\varepsilon_i) + (B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)\right), \end{aligned}$$

where $span_B$ denotes the span in the new variable B. In the rest of the proof we fix our attention on the polynomial

$$p(B) = \prod_{i=1}^n((B^{-1}+1+B)B^{a_i}+\varepsilon_i) + (B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)$$

and calculate its span in the variable B. We denote by h_F the highest (respectively by l_F the lowest) degree of $\prod_{i=1}^n((B^{-1}+1+B)B^{a_i}+\varepsilon_i)$ and let h_S be the highest (respectively l_S be the lowest) degree of $(B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)$. Here the subscripts F and S refer respectively to the first and second summands of the polynomial $p(B)$. Let h be the highest (respectively l be the lowest) degree of $p(B)$. By definition $span_B(p(B)) = h-l$, and clearly $h = \max\{h_F, h_S\}$ if $h_F\neq h_S$ and $l=\min\{l_F, l_S\}$ if $l_F\neq l_S$. The strategy will be then to calculate h_F, l_F, h_S and l_S, and whenever $h_F = h_S$ to carefully look at the possible cancellations of the highest degree terms of the summands $\prod_{i=1}^n((B^{-1}+1+B)B^{a_i}+\varepsilon_i)$ and $(B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)$. Under the hypothesis of the theorem l_F and l_S will be found to be different.

Suppose that $z>0$. Then

$$p(B) = \prod_{i=1}^n((B^{-1}+1+B)B^{a_i}+\varepsilon_i)$$

and hence

$$\begin{aligned} span_B(p(B)) &= \sum_{i=1}^n span_B((B^{-1}+1+B)B^{a_i}+\varepsilon_i)\\ &= \sum_{|a_i|>1}(|a_i|+1) + \sum_{|a_i|=1}1 + \sum_{|a_i|=0}2\\ &= \sum_{|a_i|>1}|a_i| + r + s + 2z + \alpha + \beta\\ &= \sum_{|a_i|>1}|a_i| + n + z \end{aligned}$$

and (i) follows.

Assume now that $z = 0$. We have the following values for h_F, l_F, h_S and l_S:

$$h_F = r + \sum_{a_l>1} a_l + 2\alpha - \beta,$$
$$l_F = -s + \sum_{a_j<-1} a_j + \alpha - 2\beta,$$
$$h_S = 1 + \sum_{a_l>1} a_l + \alpha,$$
$$l_S = -1 + \sum_{a_j<-1} a_j - \beta.$$

It follows then that

$$h = \begin{cases} r + \sum_{a_l>1} a_l + 2\alpha - \beta & \text{if} \quad r+\lambda > 1 \\ 1 + \sum_{a_l>1} a_l + \alpha & \text{if} \quad r+\lambda < 1 \end{cases}$$

and

$$l = \begin{cases} -s + \sum_{a_j<-1} a_j + \alpha - 2\beta & \text{if} \quad s-\lambda > 1 \\ -1 + \sum_{a_j<-1} a_j - \beta & \text{if} \quad s-\lambda < 1 \end{cases}$$

and (ii) follows from these equalities ($r+\lambda < 1$ and $s-\lambda < 1$ are not possible simultaneously).

We now prove (iii). Assume $r+\lambda = 1$, $r > 1$ and $s-\lambda \neq 1$. By (1) of Remark 4 we can suppose that $a_l > 1$, $l = 1, \ldots, r$ and $a_j < -1$, $j = r+1, \ldots, r+s$. Note first that $s-\lambda > 1$ and hence $l = -s + \sum_{a_j<-1} a_j + \alpha - 2\beta$. On the other hand we have that $h_F = h_S$, the highest degree term of $\prod_{i=1}^n ((B^{-1}+1+B)B^{a_i} + \varepsilon_i)$ is $\varepsilon_{r+1}\cdots\varepsilon_{r+s}B^{h_F}$ and the highest degree term of $(B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i} + \varepsilon_i)$ is $(-1)^r\varepsilon_{r+1}\cdots\varepsilon_{r+s}(-1)^\alpha(-1)^\beta B^{h_S}$. Since $r+\lambda = 1$, these terms cancel in $p(B)$. We then have to look at the second highest degree term. For the summand $\prod_{i=1}^n((B^{-1} +1+B)B^{a_i} + \varepsilon_i)$ this term is

$$\left((r+\alpha+\beta)\varepsilon_{r+1}\cdots\varepsilon_{r+s} + \sum_{a_k=-2} \varepsilon_{r+1}\cdots\varepsilon_{r+s}\right) B^{h_F-1} = (r+\alpha+\beta+d)\varepsilon_{r+1}\cdots\varepsilon_{r+s}B^{h_F-1}$$

where d is the number of entries $a_k = -2$ (note that $\varepsilon_k = +1$ if $a_k = -2$), and for $(B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)$ this term is

$$\Big((-1)^r(-1)^\alpha(-1)^\beta\varepsilon_{r+1}\cdots\varepsilon_{r+s} + (\alpha+\beta)(-1)^r(-1)^\alpha(-1)^\beta\varepsilon_{r+1}\cdots\varepsilon_{r+s}\Big)B^{h_F-1}$$
$$= -(\alpha+\beta+1)\varepsilon_{r+1}\cdots\varepsilon_{r+s}B^{h_F-1} \quad (\text{as } r+\lambda = 1).$$

Since $r > 1$, $(r-1+d)\varepsilon_{r+1}\cdots\varepsilon_{r+s}B^{h_F-1}$ is the highest degree term of $p(B)$. Hence $h = r + \sum_{a_l>1} a_l + 2\alpha - \beta - 1$ and (iii) follows.

Finally we prove (iv). Since $r+\lambda = 1$, $r = 1$ and $s > 1$ it follows that $s - \lambda > 1$ and so $l = -s + \sum_{a_j<-1} a_j + \alpha - 2\beta$. But as in (iii), we have that $h_F = h_S$ and the terms with this degree cancel. Moreover, other previous terms cancel, too. The question is: "How many steps do we have to go down in order to find the first no cancellation?" Recall that $\varepsilon_i = (-1)^{a_i}$ for every $i \in \{1, \ldots, n\}$, $a_1 > 1$, $a_j < -1$ if $j \in \{2, ..., n\}$ and $|a_2| \leq \ldots \leq |a_n|$. We have to add

$$\prod_{i=1}^n((B^{-1}+1+B)B^{a_i} + \varepsilon_i) = (\varepsilon_1 + B^{a_1-1} + B^{a_1} + B^{a_1+1}) \cdot \prod_{j=2}^n (B^{a_j-1} + B^{a_j} + B^{a_j+1} + \varepsilon_j)$$

and

$$(B^{-1}+1+B)\prod_{i=1}^n(-B^{a_i}+\varepsilon_i)$$
$$= (B^{-1}+1+B)(\varepsilon_1 - B^{a_1})\prod_{j=2}^n(-B^{a_j}+\varepsilon_j)$$
$$= (\varepsilon_1 B^{-1} + \varepsilon_1 + \varepsilon_1 B - B^{a_1-1} - B^{a_1} - B^{a_1+1}) \cdot \prod_{j=2}^n(-B^{a_j}+\varepsilon_j).$$

It follows (consider the case $a_1 = 2$ separately) that the highest degree h of $p(B)$ is (recall that $h_F = h_S = 1 + \sum_{a_l>1} a_l + \alpha$):

$h_F - a_1$ if $a_1 < |a_2| - 1$,

$h_F - (|a_2| - 1)$ if $a_1 > |a_2| - 1$,

$h_F - |a_2|$ if $a_1 = |a_2| - 1$ and $|a_2| < |a_3| - 1$,

$h_F - a_1$ if $a_1 = |a_2| - 1$ and $|a_2| = |a_3|$,

$h_F - (|a_3| + 1)$ if $a_1 = |a_2| - 1$, $|a_2| = |a_3| - 1$ and $|a_3| < |a_4| - 1$,

$h_F - |a_2|$ if $a_1 = |a_2| - 1$, $|a_2| = |a_3| - 1$ and $|a_3| = |a_4|$,

$h_F - |a_3|$ if $a_1 = |a_2| - 1$, $|a_2| = |a_3| - 1$ and $|a_3| = |a_4| - 1$

and (iv) follows from just rewriting the difference $h - l$. □

Remark 5. Remark 4 and Theorem 2 together provide the value of the span of the Jones polynomial $V(L)$ of any oriented pretzel link L except for the following cases:

- $P(a)$ with $a > 1$. Then $\langle P(a)\rangle$ has span zero (see Remark 1).
- $P(a,b)$ with $a > 1$ and $b < -1$ (but then $P(a,b) = P(a-1, b+1)$).
- $P(b_1, \ldots, b_s, 1)$ with $s > 1$ and $b_j < -1$, $j = 1, \ldots, s$.

Then we have that

$$p(B) = (B + B^2) \prod_{j=1}^{s} (B^{b_j - 1} + B^{b_j} + B^{b_j+1} + \varepsilon_j)$$
$$+ (B^{-1} + 1 + B)(-1 - B) \prod_{j=1}^{s} (-B^{b_j} + \varepsilon_j).$$

Assume that there is no $j \in \{1, \ldots, s\}$ with $b_j = -2$ (if for example $b_s = -2$, then $P(b_1, \ldots, b_{s-1}, b_s, 1) = P(b_1, \ldots, b_{s-1}, 2)$). Then $l = 1 - s + b_1 + \ldots + b_s$ if $s \neq 2$, $l = b_1 + b_2$ if $s = 2$ and $h = 1$ in any case, and it follows that $span(\langle P(b_1, \ldots, b_s, 1)\rangle)$ is $4(-b_1 \ldots - b_s - 1)$ if $s \neq 2$ and it is $4(-b_1 - b_2 - 2)$ if $s = 2$.

- $P(a, -a - 1, -a - 2)$ with $a > 1$. By expanding $p(B)$ in Theorem 2 we find that

$$span(\langle P(a, -a - 1, -a - 2)\rangle) = 8a \text{ if } a \neq 2$$

and

$$span(\langle P(2, -3, -4)\rangle) = 12.$$

Remark 6. Denote by $V_1(L)$ the Jones polynomial of the oriented link L with normalization $V_1(L) = 1$. Recall that $V_1(L) = (-A)^{-3w(D)} \langle D \rangle_1$ after the substitution $A = t^{-1/4}$, where D is an oriented diagram of L and $w(D)$ is its writhe. It follows that $V(L) = (-t^{-1/2} - t^{1/2}) V_1(L)$ for every oriented link L, and therefore $span(V(L)) = span(V_1(L)) + 1$.

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4. REFERENCES

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